Journal's outer Jacket ?
ISSN No ?
Link ?
Publisher's Name

Jnanabha – UGC Approved
15

Jñānābha, Vol. 51(1) (2021), 1-10

ON SIMPLE BOUNDS FOR INVERSE HYPERBOLIC SINE AND INVERSE HYPERBOLIC TANGENT FUNCTIONS

By

Ramkrishna M. Dhaigude, Sumedh B. Thool

Department of Mathematics, Government Vidarbha Institute
of Science and Humanities, Amravati-444604, Maharashtra, India
Email:rmdhaigude@gmail.com, sumedhmaths@gmail.com

Yogesh J. Bagul*

Department of Mathematics, K. K. M. College, Manwath,
Dist: Parbhani-431505, Maharashtra, India
Email:yjbagul@gmail.com

and

Vinay M. Raut

Department of Mathematics, Shri. Shivaji Science College,
Amravati-444603, Maharashtra, India
Email:vinayraut18@gmail.com

(Received : April 04, 2021 ; Revised : April 22, 2021)

Abstract

We obtain simple algebraic bounds of inverse hyperbolic sine and inverse hyperbolic
tangent functions i.e., $\sinh^{-1} x$ and $\tanh^{-1} x$. The inequalities are obtained on the entire domains
of these functions. From our results, we obtain tighter bounds for the same functions. The
Wilker and Huygens type inequalities involving inverse hyperbolic functions can also be easily
derived from our main results.

2010 Mathematics Subject Classifications: 26D07, 26D20, 42A10

Keywords and phrases: Inverse hyperbolic sine function, inverse hyperbolic tangent function,
Wilker and Huygens type inequalities, increasing-decreasing functions.

1 Introduction

The bounds for inverse hyperbolic sine and inverse hyperbolic tangent functions can be useful in applied mathematics where these functions occur frequently. The inequalities

$$(1.1)\quad \frac{1}{\sqrt{1+x^2}} < \frac{\sinh^{-1} x}{x};\ x > 0$$

and

$$(1.2)\quad \frac{\tanh^{-1} x}{x} < \frac{1}{1-x^2};\ x \in (0,1)$$

can be proved easily by Mean Value Theorem(*MVT*) [5]. For the trigonometric inequalities analogous to (1.1) and (1.2) we refer to [3]. In 2008, Zhu [16] established for $0 \le x \le r$ and $r > 0$ that

$$(1.3)\quad \frac{3}{2+\sqrt{1+x^2}} \le \frac{\sinh^{-1} x}{x} \le \frac{b+1}{b+\sqrt{1+x^2}},$$

where $b = \frac{\sqrt{1+x^2}\sinh^{-1}x - x}{x - \sinh^{-1}x}$. The inequality (1.3) is Shafer-Fink type inequality for inverse hyperbolic sine. In the same year, Zhu [17] presented another simple proof of Shafer's inequality [12, 13, 14]

$$(1.4)\quad \frac{\tanh^{-1} x}{x} < \frac{8}{3 + \sqrt{25 - \frac{80}{3}x^2}};\ 0 < x < \sqrt{15}/4.$$

Later, it is pointed out in [4] that a simple concise proof of inequality (1.4) in [17] contains a small mistake and other simple proofs of (1.4) are provided in [4]. The classic inequalities (1.3)(the right one) and (1.4) are not true on the entire domains of the respective functions i.e., on the domains of $\sinh^{-1} x$ and $\tanh^{-1} x$ as they are respectively defined on $(0, \infty)$ and $(0, 1)$. In this paper, we obtain algebraic bounds for these functions on their entire domains. New inequalities are simple and comparable with (1.3) and (1.4) as well as they give Wilker [15] and Huygens [10] type inequalities for inverse hyperbolic functions.

Before proceeding further, we recall the following two Lemmas regarding the monotonicity of functions. These Lemmas will be used to prove our main results in the next section.

***Lemma* 1.1** ([1, p.10]) *Let $f, g : [m, n] \to \mathbb{R}$ be two continuous functions which are differentiable on (m, n) and $g' \neq 0$ in (m, n). If f'/g' is increasing (or decreasing) on (m, n), then the functions $\frac{f(x)-f(m)}{g(x)-g(m)}$ and $\frac{f(x)-f(n)}{g(x)-g(n)}$ are also increasing (or decreasing) on (m, n). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

***Lemma* 1.2** ([9]) *Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ be convergent for $|x| < R$, where a_k and b_k are real numbers for $k = 0, 1, 2, \cdots$ such that $b_k > 0$. If the sequence a_k/b_k is strictly increasing(or decreasing), then the function $A(x)/B(x)$ is also strictly increasing(or decreasing) on $(0, R)$.*

2 Main results

In our first result, we extend inequality (1.1) to the right. the first result is stated as

***Theorem* 2.1** *The best possible constants α and β such that*

$$(2.1)\quad \left(\frac{1}{1+x^2}\right)^{\alpha} < \frac{\sinh^{-1} x}{x} < \left(\frac{1}{1+x^2}\right)^{\beta},\quad x \in (0, \infty)$$

are 1/2 and 1/6 respectively.

Proof. Let

$$f(x) = \frac{\ln\left(\frac{x}{\sinh^{-1} x}\right)}{\ln(1 + x^2)},\quad x \in (0, \infty)$$

and $\sinh^{-1} x = t$. Then $x = \sinh t$ for $t \in (0, \infty)$ and $f(x) = F(t)$ where

$$F(t) = \frac{\ln\left(\frac{\sinh t}{t}\right)}{\ln\left(1 + \sinh^2 t\right)} := \frac{F_1(t)}{F_2(t)}.$$

Differentiation yields

$$\frac{F_1'(t)}{F_2'(t)} = \frac{1 + \sinh^2 t}{2 \sinh t \cosh t} \cdot \frac{t \cosh t - \sinh t}{t \sinh t}$$

$$= \frac{\cosh t(t\cosh t - \sinh t)}{2t\sinh^2 t} = \frac{1}{2}\cdot\frac{t(1+\cosh 2t) - \sinh 2t}{t(\cosh 2t - 1)}$$

$$= \frac{1}{2}\cdot\frac{t\cosh 2t - \sinh 2t + t}{t\cosh 2t - t}$$

$$= \frac{1}{2}\cdot\frac{\sum_{k=0}^{\infty}\frac{2^{2k}t^{2k+1}}{(2k)!} - \sum_{k=0}^{\infty}\frac{2^{2k+1}t^{2k+1}}{(2k+1)!} + t}{\sum_{k=0}^{\infty}\frac{2^{2k}t^{2k+1}}{(2k)!} - t}$$

$$= \frac{1}{2}\cdot\frac{\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}\left(1 - \frac{2}{2k+1}\right)t^{2k+1}}{\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}t^{2k+1}}$$

$$= \frac{1}{2}\cdot\frac{\sum_{k=1}^{\infty}a_k t^{2k+1}}{\sum_{k=1}^{\infty}b_k t^{2k+1}} := \frac{1}{2}\frac{A(t)}{B(t)}.$$

Therefore,

$$\frac{a_k}{b_k} = 1 - \frac{2}{2k+1},\quad k \in \mathbb{N}.$$

Clearly, $\{a_k/b_k\}_{k=1}^{k=\infty}$ is a strictly increasing sequence. By *Lemma* **1.2**, F_1'/F_2' is strictly increasing on $(0,\infty)$. By *Lemma* **1.1**, $F(t)$ and hence $f(x)$ is a strictly increasing function on $(0,\infty)$. So

$$f(0+) < f(x) < f(\infty-).$$

By l'Hôpital's rule, we find the limits $f(0+) = 1/6$ and $f(\infty-) = 1/2$ which give double inequality (2.1).

Next, we improve left inequality of (2.1) or inequality (1.1).

***Theorem* 2.2** *For $x > 0$, the constant $a = 1/3$ such that*

$$(2.2)\qquad \frac{1}{\sqrt{1+ax^2}} < \frac{\sinh^{-1} x}{x}$$

is the best possible.

Proof. Let

$$g(x) = \frac{\left(\frac{x}{\sinh^{-1} x}\right)^2 - 1}{x^2},\quad x > 0$$

and $\sinh^{-1} x = t$. Then $x = \sinh t$ for $t \in (0,\infty)$ and $g(x) = G(t)$ where

$$G(t) = \frac{\left(\frac{\sinh t}{t}\right)^2 - 1}{\sinh^2 t} = \frac{\sinh^2 t - t^2}{t^2\sinh^2 t}$$

$$= \frac{\cosh 2t - 2t^2 - 1}{t^2\cosh 2t - t^2}$$

$$= \frac{\sum_{k=1}^{\infty}\frac{2^{2k}t^{2k}}{(2k)!} - 2t^2}{\sum_{k=0}^{\infty}\frac{2^{2k}t^{2k+2}}{(2k)!} - t^2} = \frac{\sum_{k=2}^{\infty}\frac{2^{2k}}{(2k)!}t^{2k}}{\sum_{k=2}^{\infty}\frac{2^{2k-2}}{(2k-2)!}t^{2k}}$$

$$= \frac{\sum_{k=2}^{\infty}a_k t^{2k}}{\sum_{k=2}^{\infty}b_k t^{2k}} := \frac{A(t)}{B(t)}.$$

Therefore,

$$\frac{a_k}{b_k} = \frac{2}{k(2k-1)},\quad k \geq 2.$$

Thus $\{a_k/b_k\}_{k=2}^{k=\infty}$ is a strictly decreasing sequence. By *Lemma* 1.2, $G(t)$ and hence $g(x)$ is decreasing on $(0,\infty)$. So $f(0+) > f(x)$. As $f(0+) = 1/3$, we get the required result.

Similarly, we extend and improve the inequality (1.2) in the following theorems.

Theorem 2.3 [illegible] *inequalities hold.*

$$(2.3)\quad \frac{2}{3-x^2} < \frac{\tanh^{-1} x}{x} < [illegible]$$

Proof. Let

$$h(x) = \frac{\left(\frac{x}{\tanh^{-1} x}\right) - 1}{x^2}, \quad x \in (0, 1).$$

We put $\tanh^{-1} x = t$ which gives $x = \tanh t$ for $t \in (0, \infty)$ and $h(x) = H(t)$ where

$$H(t) = \frac{\left(\frac{\tanh t}{t}\right) - 1}{\tanh^2 t} = \frac{\sinh t \cosh t - t\cosh^2 t}{t \sinh^2 t}$$

$$= \frac{\sinh 2t - t\cosh 2t - t}{t\cosh 2t - t}$$

$$= \frac{\sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} t^{2k+1} - t\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} t^{2k} - t}{t\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} t^{2k} - t}$$

$$= [illegible]$$

$$= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}\left(\frac{2}{2k+1} - 1\right) t^{2k+1}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} t^{2k+1}}$$

$$= \frac{\sum_{k=1}^{\infty} l_k t^{2k+1}}{\sum_{k=1}^{\infty} m_k t^{2k+1}} = \frac{L(t)}{M(t)}$$

Then

$$\frac{l_k}{m_k} = \frac{2}{2k+1} - 1, \quad k \geq 1.$$

Since a sequence $\{l_k/m_k\}_{k=1}^{\infty}$ is strictly decreasing we get by *Lemma* 1.2 that a function $H(t)$ is strictly decreasing on $(0, \infty)$ or that $h(x)$ is strictly decreasing on $(0, 1)$. Hence

$$h(0+) > h(x) > h(1-).$$

We finish the proof by noticing that the limits $h(0+) = -1/3$ and $h(1-) = -1$ can be easily found by l'Hôpital's rule.

Theorem 2.4 For $x \in (0, 1)$ the inequality

$$(2.4)\quad \frac{\tanh^{-1} x}{x} < \left(\frac{1}{1-x^2}\right)^{1/3}$$

is true.

Proof. Let

$$p(x) = \frac{\ln\left(\frac{\tanh^{-1} x}{x}\right)}{\ln\left(\frac{1}{1-x^2}\right)}, \quad x \in (0, 1).$$

Using the substitution [illegible] $\in (0, \infty)$ we get $p(x) = P(t)$ where

$$P(t) = \frac{\ln\left(\frac{t}{\tanh t}\right)}{\ln\left(\frac{1}{1-\tanh^2 t}\right)}$$

$$= \frac{\ln\left(\frac{\sinh t}{t}\right)}{2\ln(\operatorname{sech} t)} := \frac{1}{2}\cdot\frac{P_1(t)}{P_2(t)}$$

After differentiating we have

$$\frac{P_1'(t)}{P_2'(t)} = \frac{\tanh t - t\operatorname{sech}^2 t}{t\tanh^2 t} = \frac{\sinh 2t - 2t}{2t\sinh^2 t}$$

$$= \frac{\sinh 2t - 2t}{t\cosh 2t - t} = \frac{\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(2k+1)!}t^{2k+1} - 2t}{\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k)!}t^{2k+1} - t}$$

$$= \frac{\sum_{k=1}^{\infty}\frac{2^{2k+1}}{(2k+1)!}t^{2k+1}}{\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}t^{2k+1}} = \frac{\sum_{k=1}^{\infty} l_k t^{2k+1}}{\sum_{k=1}^{\infty} m_k t^{2k+1}} := \frac{L(t)}{M(t)}.$$

From this we obtain

$$\frac{l_k}{m_k} = \frac{2}{2k+1},\ k \in \mathbb{N}$$

and easily see that the sequence $\{l_k/m_k\}_{k=1}^{\infty}$ is strictly decreasing. By subsequent use of ***Lemmas* 1.1** and **1.2**, we conclude that the function $P(t)$ is strictly decreasing on $(0,\infty)$ or equivalently the function $p(x)$ is strictly decreasing on $(0,1)$. Hence $p(0-) = 1/3 > p(x)$. This completes the proof.

Now we state and prove one lemma for establishing asymptotically better lower bound for $(\sinh^{-1} x)/x$.

***Lemma* 2.1** *The inequalities*

$$\left(\frac{\sinh x}{x}\right)^2 < \cosh x,\quad x \in (0,\delta)$$

and

$$\left(\frac{\sinh x}{x}\right)^2 > \cosh x,\quad x \in (\delta,\infty)$$

hold for $\delta \approx 2.675$.

Proof. Let

$$t(x) = \sinh^2 x - x^2\cosh x,\quad x \in (0,\infty).$$

Using known series expansions we write

$$t(x) = \frac{1}{2}(\cosh 2x - 1) - x^2\cosh x$$

$$= -\frac{1}{2} + \sum_{k=0}^{\infty}\frac{2^{2k-1}}{(2k)!}x^{2k} - \sum_{k=0}^{\infty}\frac{x^{2k+2}}{(2k)!}$$

$$= \sum_{k=2}^{\infty}\frac{1}{(2k-2)!}\left[\frac{2^{2k-1}}{2k(2k-1)} - 1\right]x^{2k}$$

$$= -\frac{x^4}{6} + \frac{x^6}{360} + \frac{x^8}{560} + \frac{211x^{10}}{1814400} + \cdots,$$

Clearly, $t(x) > 0$ for all $x \geq 3$. From the nature of curves $\left(\frac{\sinh x}{x}\right)^2$ and $\cosh x$, they can intersect each other in at most one point. So the equation $t(x) = 0$ has a unique solution that lies in $(0,3)$ and can be confirmed by using bisection method[2] as follows.

5

Let $y_1 = \sinh^2 x$ [illegible] and $y_2 = x^2 \cosh x$. Then with the help of tables in [6], we have

Table 2.1

x(rad.)	y_1	y_2	$f(x) = y_1 - y_2$
1.0	1.3811	1.54308	-0.16198 < 0
3.0	100.3578	90.60894	+9.74886 > 0

From *Table* 2.1, the root of $f(x)$ lies between 1 and 3.

Table 2.2

x(rad.)	y_1	y_2	$f(x) = y_1 - y_2$
1.0	1.3811	1.54308	-0.16198 < 0
1.5	4.03383	5.29292	-1.25909 < 0
2.0	13.1541	15.0488	-1.8947 < 0
2.5	36.60495	38.32681	-1.72186 < 0
3.0	100.3578	90.60894	+9.74886 > 0

From *Table* 2.2, the root of $f(x)$ lies between 2.5 and 3.0.

Table 2.3

x(rad.)	y_1	y_2	$f(x) = y_1 - y_2$
2.5	36.60495	38.32681	-1.72186 < 0
2.6	44.81945	45.75851	-0.93906 < 0
2.7	54.85275	54.4816	+0.37115 > 0
2.8	67.1075	64.7014	+2.4061 > 0
2.9	82.07565	76.65362	+5.42203 > 0
3.0	100.3578	90.60894	+9.74886 > 0

From *Table* 2.3, the root of $f(x)$ lies between 2.6 and 2.7.

Table 2.4

x(rad.)	y_1	y_2	$t(x) = y_1 - y_2$
2.60	44.81945	45.75851	-0.93906 < 0
2.61	45.7349	46.56949	-0.83459 < 0
2.62	46.66885	47.3936	-0.72475 < 0
2.63	47.62165	48.23099	-0.60934 < 0
2.64	48.59375	49.08188	-0.48813 < 0
2.65	49.58545	49.94641	-0.36096 < 0
2.66	50.5972	50.82488	-0.22768 < 0
2.67	51.6294	51.71738	-0.08798 < 0
2.68	52.6824	52.62422	+0.05818 > 0
2.69	53.7567	53.54555	+0.21115 > 0
2.70	54.85275	54.4816	+0.37115 > 0

From *Table* 2.4, the root of $t(x)$ lies between 2.67 and 2.68. Thus the solution δ of $t(x) = 0$ can be taken as ≈ 2.675. Moreover, $t(x) < 0$ in $(0, \delta)$ and $t(x) > 0$ in (δ, ∞).

***Theorem* 2.5** *The inequality*

$$(2.5) \quad \frac{1}{1+\lambda x} < \frac{\sinh^{-1} x}{x}, \quad x \in (0, \infty)$$

holds with the best possible constant $\lambda = \frac{\sinh \delta - \delta}{\delta \sinh \delta} \approx 0.235361$ *where* δ *is the unique solution of the equation* $\sinh^2 x - x^2 \cosh x = 0$ *as given in* ***Lemma*** **2.1**.

Proof. Let

$$\varphi(x) = \frac{x - \sinh^{-1} x}{x \sinh^{-1} x}, \quad x \in (0, \infty)$$

and $\sinh^{-1} x = t$. Then $x = \sinh t$, $t \in (0, \infty)$ and $\varphi(x) = \phi(t)$ where

$$\phi(t) = \frac{\sinh t - t}{t \sinh t} = \frac{1}{t} - \frac{1}{\sinh t}.$$

We differentiate $\phi(t)$ with respect to t and get

$$\phi'(t) = -\frac{1}{t^2} + \frac{\cosh t}{\sinh^2 t}, \quad t \in (0, \infty).$$

By *Lemma* **2.1**, we infer that $\phi'(t) > 0$ for $(0, \delta)$ and $\phi'(t) < 0$ for (δ, ∞). Thus $\phi(t)$ is increasing in $(0, \delta)$ and decreasing in (δ, ∞). This implies that $\varphi(x)$ is increasing in $(0, \sinh \delta)$ and decreasing in $(\sinh \delta, \infty)$. So $\varphi(\sinh \delta) > \varphi(x)$ in either case and we get the inequality (2.5).

With the help of any graphical software, it can be seen that the lower bound of $(\sinh^{-1} x)/x$ in (2.5) is better than the corresponding lower bound in (1.3) if $x \geq \zeta$ where $\zeta \approx 2.8162$ i.e., the inequality (2.5) is asymptotically far better. Again the upper bound in (2.1) is stronger than the corresponding upper bound in (1.3) if $r \to \infty$. In a similar sense, inequality (2.4) is better than inequality (1.4).

3 Applications

In this section, we present some applications of our results obtained in section 2. First of all, we again obtain new bounds for $\sinh^{-1} x$ and $\tanh^{-1} x$ which are tighter than those obtained in section 2.

Proposition **3.1** *If $x \in (0, \infty)$, then we have*

(3.1) $\frac{1}{x}\left(3\sqrt{1+x^2/3} - \sqrt{1+x^2} - 4\right) < \sinh^{-1} x$

and

(3.2) $\frac{1}{\lambda x}\left(\lambda x - \ln(1+\lambda x) + \lambda^2\left(\sqrt{1+x^2} - 1\right)\right) < \sinh^{-1} x$

where λ is as defined in Theorem **2.5**.

Proof. From inequality (2.2), we write

$$\int_0^x \frac{t}{\sqrt{1+t^2}}\,dt < \int_0^x \sinh^{-1} t \cdot dt, \; t \in (0, x) \text{ and } x \in (0, \infty).$$

After evaluating we get

$$3\left[\sqrt{1+t^2/3}\right]_0^x < \left[t \sinh^{-1} t - \sqrt{1+t^2}\right]_0^x.$$

Equivalently,

$$3\sqrt{1+x^2/3} - \sqrt{1+x^2} - 4 < x \sinh^{-1} x.$$

This gives inequality (3.1) and inequality (3.2) is obtained by a similar technique using (2.5).

It is observed that the inequality (3.1) is tighter than (2.2). The inequality (3.2) is sharper than (2.5) in $(0, \gamma)$ where $\gamma \approx 2.874$ and it is also sharper than the left inequality of (1.3) in the large interval (ξ, ∞) where $\xi \approx 2.7855$.

Proposition 3.2 *The inequalities*

(3.3) $\frac{1}{x}\ln\left(\frac{1}{(1-x^2/3)^{3/2}\sqrt{1-x^2}}\right) < \tanh^{-1} x$

$$< \frac{1}{2x}\left[\ln\left(\frac{1}{1-x^2}\right) - \frac{3}{2}\left(\left(1-x^2\right)^{2/3} - 1\right)\right]$$

hold in $(0, 1)$.

Proof. Consider the combination of left inequality of (2.3) and the inequality (2.4) as

$$\frac{3}{3-t^2} < \frac{\tanh^{-1} t}{t} < \left(\frac{1}{1-t^2}\right)^{1/3}, \; t \in (0, x).$$

Then integrating this inequality over $(0, x)$, we get desired inequalities (3.3).

The left inequality of (3.3) is sharper than the corresponding left inequality of (2.3). Again the right inequality of (3.3) is sharper than (2.4) and it is also tighter than (1.4) if $x \geq \zeta \approx 0.8595$.

Secondly, the inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \; x \in (0, \pi/2)$$

and

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3, \; x \in (0, \pi/2)$$

were proposed respectively by J. B. Wilker [15] and C. Huygens [10]. Here, we state and prove the inequalities of the above type for inverse hyperbolic functions $(\sinh^{-1} x)/x$ and $(\tanh^{-1} x)/x$.

Proposition 3.3 *(Wilker type inequality). For $x \in (0, 1)$, the inequality*

(3.4) $$\left(\frac{\sinh^{-1} x}{x}\right)^2 + \frac{\tanh^{-1} x}{x} > 2$$

holds true.

Proof. From (2.2) and the left inequality of (2.3) we have

$$\left(\frac{\sinh^{-1} x}{x}\right)^2 + \frac{\tanh^{-1} x}{x} > \frac{3}{3+x^2} + \frac{3}{3-x^2}$$
$$= \frac{18}{(3+x^2)(3-x^2)}$$
$$= 2 \cdot \frac{9}{9-x^4} > 2.$$

The proof is complete.

Proposition 3.4 *(Huygens type inequality): For $x \in (0, 1)$, it is true that*

(3.5) $$2\frac{\sinh^{-1} x}{x} + \frac{\tanh^{-1} x}{x} > 3.$$

Proof. Again from the inequalities (2.2) and (2.3) we have

$$2\frac{\sinh^{-1} x}{x} + \frac{\tanh^{-1} x}{x} > \frac{2\sqrt{3}}{\sqrt{3+x^2}} + \frac{3}{3-x^2}, \quad x \in (0, 1).$$

The inequality (3.2) will follow if we prove that

$$\frac{2\sqrt{3}}{\sqrt{3+x^2}} + \frac{3}{3-x^2} > 3$$

i.e.

$$\frac{2}{\sqrt{3(3+x^2)}} + \frac{1}{3-x^2} > 1$$

or

$$\sqrt{3(3+x^2)}x^2 - 2\sqrt{3(3+x^2)} > 2x^2 - 6.$$

Since the quantities at both sides of the last inequality are negative on (0, 1), after squaring and simplifying we say that we want

$$33x^6 + 211x^4 + 288x^2 > 0$$

which is true on (0, 1).

For refinements of inequalities (3.4) and (3.5) we refer to [7].

Acknowledgement. We are very much thankful to the Editor and Reviewer for their suggestions to improve the version in its present form.

9

References

[1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Maps*, John Wiley and Sons, New York, 1997.

[2] K. E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley and Sons, New York, 2nd edn., 1991.

[3] Y. J. Bagul, and R. M. Dhaigude, Simple efficient bounds for arcsine and arctangent functions, (submitted), *South East Asian Journal of Mathematics and Mathematical Sciences*, 2021.

[4] Y. J. Bagul, R. M. Dhaigude, Alternative proofs of Shafer's inequality for inverse hyperbolic tangent, Preprint, arXiv:2103.14037, 2021.

[5] R. G. Bartle, and D. R. Sherbert, *Introduction to Real Analysis*, John Wiley and Sons(Asia), Singapore, Third ed., 2007

[6] G. F. Becker, and C. E. Van Orstrand, *Smithsonian Mathematical Tables: Hyperbolic Functions*, Smithsonian Institution, Washington, 1909.

[7] C.-P. Chen, Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions, *Integral Transforms and Special Functions*, 23(12), (2012), 865-873. Doi: 10.1080/10652469.2011.644851

[8] A. M. Fink, Two inequalities, *Univ. Beograd. Pub. Elektrotehn. Fak. Ser. Mat.* 6(1995), (1995) 48-49.

[9] V. Heikkala, M. K. Vamanamurthy and M. Vuorinen, Generalized elliptic integrals, *Comput. Methods Funct. Theory*, 9(1) (2009), 75-109.

[10] C. Huygens, *Oeuvres Completes*, Société Hollandaise des Sciences, Haga, 1888-1940.

[11] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[12] R. E. Shafer, Elementary Problems: E 1867, *The American Mathematical Monthly*, 73(3) (1966), 309.

[13] R. E. Shafer, Analytic inequalities obtained by quadratic approximation, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 577-No. 598 (1977), 96-97.

[14] R. E. Shafer, On quadratic approximation II, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 602-No. 603 (1978), 163-170.

[15] J. B. Wilker, Elementary Problems: E3301-E3306, *Amer. Math. Monthly*, 96(1) (1989), 54-55.

[16] L. Zhu, New inequalities of Shafer-Fink type for arc hyperbolic sine, *Journal of Inequalities and Applications*, 2008, Article ID 3682275, (2008), 5. Doi: 10.1155/2008/368275

[17] L. Zhu, On a quadratic estimate of Shafer, *J. Math. Ineqal.*, 2(4), (2008), 571-574. Doi: 10.7153/jmi-02-51

Juni Khyat
(UGC Care Group I Listed Journal)
ISSN: 2278-4632
Vol-11 Issue-02 No.02 February 2021

TIME FRACTIONAL MAGNETO-THERMOELASTICITY WITH ROSENTHAL HEAT SOURCE AND EDDY CURRENT LOSS

J.J.Bikram & G.D.Kedar Department of Mathematics, RTM Nagpur University, Nagpur-440 033 (Maharashtra) India.

L.C.Bawankar, Department of Mathematics, Sanjivani College of Engineering, Kopargaon, Ahmednagar- 423 603 (Maharashtra), India.

V.M.Raut Department of Mathematics, Shri Shivaji Science College, Amravati-444603, India.

Abstract—In this paper, the analytical solution is obtained for time fraction heat conduction equation and thermal stresses in one dimensional finite rod induced by transient magnetic field. Heat generation is considered as a combination of Rosenthal heat source and Eddy current loss. The equations of electromagnetic field, heat conduction and elastic field are formulated. The Laplace and Fourier transform techniques are adopted [illegible]. The stresses are obtained in terms of sum of thermal and magnetic stress components. The thermal and magnetic stress components are arises due to eddy current loss and Lorentz force respectively. The time dependent magnetic field is taken in the form of exponential profile for numerical calculation. The results are illustrated graphically to understand the effect of fractional order parameter on magnetic field, eddy current loss, temperature distribution and stresses.

Keywords— Fractional thermoelasticity, Magneto-thermoelastic, Lorentz force, Eddy current, Rosenthal heat source, Thermal stresses.

Introduction

Many researchers studied the problems of thermoelasticity under the influence of magnetic field. We consider time dependent magnetic field if it acts on a conducting medium, generates heat which results into eddy current loss, causes temperature change in body. Subsequently two kinds of stresses emerge in the body; one is thermal stress due to eddy current loss and magnetic stress due to Lorentz force. The quasi-static problems with transient thermal stresses in thin and thick bodies with and without heat generation were studied by many authors. Kulkarni et al. [1] explained the quasi-static thermal stress problem of rectangular plate subjected to constant heat supply. Deshmukh and Khandait [2] discussed about the thermal stress in a simply supported rectangular plate. Stoll [3] has explained that by changing magnetic field generates eddy current which are loops of electrical conductor. The behavior of stress under the influence of magnetic field was explained by [4]. [illegible] et al. [5] have discussed the effect of transient magnetic field on thermoelastic stresses in a conducting plate and determined quasi-static stresses in a one dimensional problem. In welding engineering problems such as cutting of metals, grinding, hardening of alloys, laser cladding, etc; the cause of thermal conduction results from a Rosenthal heat source. It takes place in the form of transient heat transfer. The idea of evaluating the temperature distribution in solids due to Rosenthal heat source introduce by many researchers [6, 7, 8]. However, these theories are restricted to quasi stationary state and integer order heat conduction equation. The fractional calculus provides a generalization of the derivatives and integration to non-integer order [9]. Past few decades, shows the remarkable contribution of fractional calculus to both experimental and theoretical field. He and Guo [10] study the effect of time, velocity of the moving heat source and fractional order parameter on considered temperature, displacement and thermal stress for one dimensional thermoelastic rod. In the theory of [11, 12], the constitutive equation with the long tail power time non-local kernel is taken

$$q(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\, grad(T)\,d\tau, \quad 0<\alpha\le 1 \qquad (1)$$

Principal

ISSN: 2278-4632
Vol-11 Issue-02 [illegible] February 2021

[illegible]

where q is the heat flux related to the temperature gradient $q = -k_t \nabla T$, k_t is the thermal conductivity. Eqns (1) and (2) leads to the time fractional heat conduction equation with Caputo derivative [13],[14]

$$\frac{1}{a}\frac{\partial^\alpha T}{\partial t^\alpha} = \Delta T, 0 < \alpha \leq 2, \tag{3}$$

where T (x,t) is temperature change, $\frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional derivative defined as [11]

$$\frac{\partial^\alpha T}{\partial t^\alpha} = D^\alpha T = \begin{cases} \frac{\partial^\alpha T}{\partial t^\alpha} & for\ \alpha = n \in R \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}d\tau & n-1 < \alpha < n \end{cases} \tag{4}$$

The equation (3) provides the whole spectrum from local heat conduction (as $\alpha \to 0$) through the standard heat conduction ($\alpha = 1$) to the ballistic heat conduction [illegible] diffusivity coefficient, ρ the mass density, C the specific heat capacity.

The aim of the work is to obtain the mathematical model for one dimensional time fractional thermoelastic field with Rosenthal heat source. The solution is obtained for temperature and elastic field in presence of magnetic field. The Laplace and Fourier transform techniques are adopted to deal with time and space variable respectively. The stresses obtained are the combination of thermal and magnetic stresses which are due to eddy current loss, Rosenthal heat source and Lorentz force respectively. For numerical calculation excitation is taken in the form of exponential function. The effect of fractional order parameter is observed on temperature distribution, thermal stresses, magnetic stresses, eddy current loss and magnetic field.

2 Mathematical Formulation

2.1 Electromagnetic Field

Consider the magnetic field $H = (0, 0, H_Z(x,t))$ in the conducting plate, and the electric field be $E = (0, E_Y(x,t), 0)$. The conducting plate of thickness 2L with time dependent magnetic field is considered as $H_0\varphi(t)$ distributed uniformly and act on both side surfaces of the plate, where H_0 be a magnetic field strength and $\varphi(t)$ is time function

[illegible] (5)

$$\text{curl} E = -\dot{B} \tag{6}$$

The governing equations of electromagnetism by neglecting the displacement current are given by [5]

$$J_y = \sigma E_y$$
$$B_z = \mu H_z \tag{8}$$

where B_z is magnetic flux in z direction, J_y is the components of the current density in y direction, μ is magnetic permeability and σ is electric conductivity in a conducting rod.
The equation of magnetic field is given by [5]

$$H_{z,xx} = \mu\sigma H_{z,t} \tag{9}$$

where , comma denotes partial differentiation with respect to followed variable. The initial and boundary conditions are given by

$$\text{at } x = \pm L; \quad H_z = H_0\varphi(t)$$
$$\text{at } t = 0; \quad H_z = 0 \tag{10}$$

The current density is denoted by J_y which is produced by change in magnetic field called eddy current which produces joule heat. The eddy current loss $\omega(x,t)$ is obtained from eddy current J_y.
The eddy current loss is given by [5]

$$\omega(x,t) = \frac{|J_y(x,t)|^2}{\sigma} \tag{11}$$

2.2 Temperature Field

In this paper, we consider homogeneous isotropic thermoelastic finite space occupying the region $-L \le x \le L$. It is assume that the initial state of the medium is quiescent and insulated at boundary. The co-ordinates (x,t) are used and the problem is restricted to moving heat source. The problem is thus one dimensional with all functions considered depending on the space variable x as well as on the time variablet. The surface of the medinm is taken traction free. The problem is considered within the context of the theory of thermoelasticity with time fractional order α.
The one dimensional time fractional order heat conduction equation with instantaneous moving heat source is [11, 15]

$$\frac{1}{a}\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial^2 T}{\partial x^2} + \frac{g(x,t)}{K}, \qquad 0 < \alpha \le 2, \quad -L < x < L, t > 0 \tag{12}$$

The boundary conditions and the initial condition are

$$\frac{\partial T(-L,t)}{\partial x} = 0$$
$$\frac{\partial T(L,t)}{\partial x} = f(t),$$
$$T(x,0) = 0, \; 0 < \alpha < 1,$$
$$T(x,0) = \left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0} = 0, \; 1 < \alpha \le 2. \tag{13}$$

where $g(x,t)$ is the heat generator factor, K is bulk modulus.

2.3 Elastic field

 ISSN: 2278-4632
Vol-11 Issue-01 No.02 February 2021

The effect of magnetic field results the occurrence of Lorentz force. For one dimensional problem it has component only [5]

$$f_x = J \times B = \begin{pmatrix} 0 \\ \frac{\partial H_z}{\partial x} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \mu H_z \end{pmatrix} = \begin{pmatrix} \frac{\mu}{2}\frac{\partial [H_z(x,t)]^2}{\partial x} \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

Simplifying the equation [illegible]

$$f_x = \frac{\mu}{2}\frac{\partial [H_z(x,t)]^2}{\partial x} \tag{15}$$

where f_x depends on x and t. The displacement vector is considered as $(u(x,t),0,0)$. The analysis of the stresses and deformations in the rod is carried out by considering the stress displacement relations defined as in [illegible]

$$\sigma_{xx}(x,t) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[\frac{\partial u}{\partial x} - \frac{1+\nu}{1-\nu}\alpha T\right]$$

$$\sigma_{yy}(x,t) = \sigma_{zz}(x,t) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[\frac{\nu}{1-\nu}\frac{\partial u}{\partial x} - \frac{1+\nu}{1-\nu}\alpha T\right] \tag{16}$$

$$\sigma_{xy}(x,t) = \frac{E}{2(1+\nu)}\frac{\partial u}{\partial x}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ the stress components and E the Young's modulus, ν the Poisson's ratio and αt the coefficient of linear thermal expansion. The [illegible] Lorentz force is given by

$$f_x + \frac{\partial \sigma_{xx}}{\partial x} = 0 \tag{17}$$

To simplify the displacement equation of motion (17) use the eqns (15) and (16) one obtains

$$\frac{\partial^2 u}{\partial x^2} = \frac{1+\nu}{1-\nu}\alpha\frac{\partial T}{\partial x} - \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\frac{\mu}{2}\frac{\partial [H_z(x,t)]^2}{\partial x} \tag{18}$$

The mechanical boundary conditions and initial condition are

$$\frac{\partial u(\pm a,t)}{\partial x} = \frac{1+\nu}{1-\nu}\alpha T,$$

$$u(x,0) = \frac{\partial u}{\partial t} = 0 \tag{19}$$

2.4 Dimensionless Quantities

The following dimensionless quantities are defined to reduce the physical components of the problem

$$\overline{T} = \frac{T}{T_0},\ \overline{x} = \frac{x}{L},\ \tau = \left(\frac{\alpha}{L}\right)^{\frac{1}{2}} t,\ [illegible],\ \overline{H} = \frac{H}{H_0},\ [illegible],\ (\overline{\sigma_{xx}}, \overline{\sigma_{yy}}) = \frac{[illegible]}{\alpha_t E T_0}(\sigma_{xx}, \sigma_{yy}),$$

$$\overline{\sigma_{ij}} = \frac{2\nu}{\alpha_t \beta E T_0}\sigma_{ij},\ \overline{u} = \frac{[illegible]}{\alpha_t L T_0}\,[illegible] \tag{20}$$

$$A = \frac{\sigma\mu l^2 a^2}{t^*}, \bar{T}_i = \frac{M}{\mu H_0^2}, \bar{\beta} = \frac{\nu(1-2\nu)}{E(1-\nu)}, Z = \frac{\mu H_0^2}{(2\alpha_t T_0)} \quad (21)$$

3 Solution

3.1 Electromagnetic Field

Introducing the dimensionless components in (9) and (10), the electromagnetic field along with boundary and initial conditions reduces to

$$\frac{\partial^2 \bar{H}_z}{\partial \bar{x}^2} = A\frac{\partial \bar{H}_z}{\partial \tau} \quad (22)$$

$$\text{at } \bar{x} = \pm 1, \ \bar{H}_z = \phi(\tau)$$
$$\text{at } \bar{\tau} = 0, \ \bar{H}_z = 0 \quad (23)$$

Eddy current is

$$\bar{J}_y(\bar{x},\tau) = \frac{\partial \bar{H}_z(\bar{x},\tau)}{\partial \bar{x}} \quad (24)$$

Dimensionless form of eddy current loss (11) is obtained as

$$\bar{Q}(\bar{x},\tau) = \left[\bar{J}_y(\bar{x},\tau)\right]^2 \quad (25)$$

The solution of electromagnetic field can be obtained by introducing the inhomogeneous boundary conditions into homogeneous, one defines

$$\bar{H}_z(\bar{x},\tau) = h_z(\bar{x},\tau) + \phi(\tau) \quad (26)$$

Substituting (26) into (22) and (23) we get

$$\frac{\partial^2 h_z}{\partial \bar{x}^2} = A\left(\frac{\partial h_z}{\partial \tau} + \frac{\partial \phi(\tau)}{\partial \tau}\right) \quad (27)$$

The boundary conditions and [illegible]

$$\text{at } \bar{x} = \pm 1, \ h_z = 0 \quad (28)$$
$$\text{at } \bar{\tau} = 0, \ h_z = -\phi(0)$$

One assumes the solution of equation (27) as

$$h_z(\bar{x},\tau) = \sum_{n=1}^{\infty} a_n(\tau)\cos(k_n \bar{x}) \quad (29)$$

where an(τ) is unknown and kn are the positive roots of eigen equation

$$\cos(k_n) = 0 \qquad k_n = \frac{(2n-1)\pi}{2}, n = 1,2, \quad (30)$$

Substitute the assumed solution (29) in equation (27) one gets,

$$\sum_{n=1}^{\infty} -k_n^2 a_n(\tau)\cos(k_n \bar{x}) = A\sum_{n=1}^{\infty}\frac{\partial a_n(\tau)}{\partial \tau}\cos(k_n \bar{x}) + A\frac{\partial \phi(\tau)}{\partial(\tau)} \quad (31)$$

Multiply equation (31) by $\cos(k_m \bar{x})$ and integrate it from −1 to 1 and using orthogonal property

$$\int_{-1}^{1}\cos(k_m \bar{x})\cos(k_n \bar{x})d\bar{x} = \begin{cases} 1 (m = n) \\ 0 (m \neq n) \end{cases} \quad (32)$$

One gets

$$\frac{\partial a_n(\tau)}{\partial\tau} + \frac{k_n^2 a_n(\tau)}{1} = \int \frac{\partial\phi(\tau)}{\partial(\tau)}\cos(k_n x)\,dx \qquad (33)$$

The solution of linear differential equation (33) is obtained as

$$a_n(\tau) = \frac{2(-1)^n}{k_n}\int_0^{\tau} e^{[illegible]}\frac{\partial\phi(\tau')}{\partial(\tau')}d\tau' \qquad (34)$$

The complete solution of the electromagnetic field (26) becomes

$$\overline{H}_z(\overline{x},\tau) = \phi(\tau) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{k_n}\cos(k_n\overline{x})b_n(\tau) \qquad (35)$$

Where $b_n(\tau) = \int_0^{\tau} e^{[illegible]}\frac{\partial\phi(\tau')}{\partial(\tau')}d\tau'$ (36)

Insert the above equation in (24) and the eddy current loss (25), we get

$$[illegible] = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin(k_n\overline{x})b_n(\tau) \qquad (37)$$

$$\omega(\overline{x},\tau) = 4\sum\sum(-1)^{[illegible]}\sin(k_m\overline{x})\sin(k_n\overline{x})b_m(\tau)b_n(\tau) \qquad (38)$$

3.2 Temperature Field

Using dimensionless quantities (20) the fractional heat conduction equation leads to

$$\frac{\partial^{\alpha}\overline{T}}{\partial\tau^{\alpha}} = \frac{\partial^2\overline{T}}{\partial\overline{x}^2} + \overline{g}(\overline{x},\tau), \quad 0<\alpha\le 2, \quad -1\le\overline{x}\le 1, \tau>0 \qquad (39)$$

The boundary conditions and the initial condition in dimensionless form are

$$\frac{\partial\overline{T}(-1,\tau)}{\partial\overline{x}} = 0,$$

$$\frac{\partial\overline{T}(1,\tau)}{\partial\overline{x}} = f_1(\tau),$$

$$\overline{T}(\overline{x},0) = 0,\ 0<\alpha\le 1,$$

$$\overline{T}(\overline{x},0) = \frac{\partial\overline{T}(\overline{x},\tau)}{\partial\tau}\Big|_{[illegible]} = 0,\ 1<\alpha\le 2 \qquad (40)$$

The heat source $\overline{g}(\overline{x},\tau)$ in (39) is a moving heat source of Rosenthal type of constant strength $g_0\,(W/m^2)$ moving with constant velocity ϑ along positive x direction, due to the effect of eddy current loss $\omega(\overline{x},\tau)$ is assumed as

$$\overline{g}(\overline{x},\tau) = g_0\,\delta(\overline{x}-\vartheta\tau)\,\omega(\overline{x},\tau). \qquad (41)$$

By employing a new co-ordinate ξ the fixed coordinate system is transformed to the moving coordinate system with the source. This is made possible by choosing:

$$\xi = \overline{x} - \vartheta\tau \qquad (42)$$

Introducing the above transformation to the non-dimensional fractional differential (39), we get dimensionless fractional heat conduction equation with Rosenthal moving heat source as

$$\frac{\partial^2 \bar{T}}{\partial \tau^2} + (-\vartheta)^2 \frac{\partial^2 \bar{T}}{\partial \xi^2} = \frac{\partial^2 \bar{T}}{\partial \xi^2} + g_0 \delta(\xi)\omega(x,\tau) \tag{43}$$

Using Laplace transform for Caputo fractional derivative with parameter s and quiescent initial condition, heat conduction equation (43) will takes the form

$$s^2 \bar{T}(\xi,s) - (-\vartheta)^{n} \frac{\partial^2 \bar{T}(\xi,s)}{\partial \xi^2} = \frac{\partial^2 \bar{T}(\xi,s)}{\partial \xi^2} + g_0\, \delta(\xi)\bar{\omega}(\xi,s) \tag{44}$$

Taking Fourier transform of equation (44) with ζ parameter, one obtained the temperature distribution in the rod as

$$\bar{T}(\zeta,s) = \frac{g_0 F\{\delta(\xi)\bar{\omega}(\xi,s)\}}{\{s^{\alpha} + \zeta^2 - (-1)^n \vartheta^2 \zeta^n\}} \tag{45}$$

Applying inverse Laplace transforms and inverse Fourier transform to (45) and then in obtained solution back substituting ξ, one gets the temperature distribution as

$$\bar{T}(x,\tau) = 4g_0 \sum\sum\sum\sum\sum (-1)^{[illegible]} \frac{e^{[illegible]} f(k_n - k_s)\sin(k_n \vartheta\tau)\sin(k_m \vartheta\tau)\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)\Gamma(\alpha(k+1)+j+2)} \tag{46}$$

$$\cos(\gamma_p(x-\vartheta\tau)) \left\{ \frac{2\sin(x-\vartheta\tau)}{[illegible]} - \frac{2\cos(x-\vartheta\tau)}{[illegible]} + (-1)^n \vartheta^n \sum \frac{(-1)^r (x-\vartheta\tau)^r}{[illegible]} \right\}$$

Where

$$f(k_n, k_s) = [illegible], \quad \gamma_p = \frac{(2p+1)\pi}{2} \tag{47}$$

3.3 Elastic field

Using (20) and (21), the dimensionless form of lorentz force (15) is

$$\bar{F} = \frac{1}{2} \frac{\partial [\bar{H}_z(x,\tau)]^2}{\partial x} \tag{48}$$

The new form of stress displacement relations (16) by using dimensionless quantities (20), (21) are

$$\bar{\sigma}_{xx}(x,\tau) = \frac{\partial \bar{u}}{\partial x} - \bar{T}$$

$$\bar{\sigma}_{\theta\theta}(x,\tau) = \beta \frac{\partial \bar{u}}{\partial x} - \bar{T} \tag{49}$$

$$\bar{\sigma}_{zz}(x,\tau) = \frac{\partial \bar{u}}{\partial x}$$

Also the dimensionless equation of motion

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\nu}{\beta(1-\nu)} \frac{\partial \bar{T}}{\partial x} - \gamma \frac{\partial [\bar{H}_z]^2}{\partial x} \tag{50}$$

with corresponding boundary conditions and initial condition are

$$\frac{\partial \bar{u}(\pm 1,\tau)}{\partial x} = \frac{\nu}{\beta(1-\nu)} \bar{T},$$

$$\bar{u}(x,0) = \frac{\partial \bar{u}}{\partial \tau} = 0 \tag{51}$$

The displacement and stresses solution are combination of thermal and magnetic components, obtained due to eddy current loss and Lorentz force as follows

Juni Khyat
(UGC Care Group I Listed Journal)
ISSN: 2278-4632
Vol-11 Issue-01 No.02 February 2021

$$\bar{u}(\bar{x},\tau)=\bar{u}^{T}(\bar{x},\tau)+\bar{u}^{M}(\bar{x},\tau) \tag{52}$$

$$\begin{aligned}\bar{\sigma}_{xx}(\bar{x},\tau)&=\bar{\sigma}_{xx}^{T}(\bar{x},\tau)+\bar{\sigma}_{xx}^{M}(\bar{x},\tau)\\ \bar{\sigma}_{yy}(\bar{x},\tau)&=\bar{\sigma}_{yy}^{T}(\bar{x},\tau)+\bar{\sigma}_{yy}^{M}(\bar{x},\tau)\\ \bar{\sigma}_{zz}(\bar{x},\tau)&=\bar{\sigma}_{zz}^{T}(\bar{x},\tau)+\bar{\sigma}_{zz}^{M}(\bar{x},\tau)\end{aligned} \tag{53}$$

where the suffix T represents thermal component arises due to ... and M magnetic component arises due to Lorentz force. The thermal and magnetic components of displacement are obtained by solving the equation (50) and using conditions (51)

$$\begin{aligned}\bar{u}^{T}(\bar{x},\tau)&=\frac{\nu}{\beta(1-\nu)}\int_{1}^{\bar{x}}\bar{T}(\bar{x},\tau)d\bar{x}\\ \bar{u}^{M}(\bar{x},\tau)&=[\varphi(\tau)]^{2}-\frac{\nu}{\beta(1-\nu)}\int_{1}^{\bar{x}}[\bar{H}_{z}]^{2}d\bar{x}\end{aligned} \tag{54}$$

Putting these displacement components (56) in equation (51), the thermal and magnetic stress components are obtained as

$$\begin{aligned}&\bar{\sigma}_{xx}^{T}(\bar{x},\tau)=0 \quad \bar{\sigma}_{xx}^{M}(\bar{x},\tau)=\frac{\nu}{\beta(1-\nu)}\left([\varphi(\tau)]^{2}-[\bar{H}_{z}]^{2}\right)\\ &\bar{\sigma}_{yy}^{T}(\bar{x},\tau)=\frac{(2\nu-1)}{(1-\nu)}\bar{T}, \quad \bar{\sigma}_{yy}^{M}(\bar{x},\tau)=\frac{\nu}{(1-\nu)}\left([\varphi(\tau)]^{2}-[\bar{H}_{z}]^{2}\right)\\ &\bar{\sigma}_{zz}^{T}(\bar{x},\tau)=\frac{\nu}{\beta(1-\nu)}\bar{T}, \bar{\sigma}_{zz}^{M}(\bar{x},\tau)=-\frac{\nu}{\beta(1-\nu)}\left([\varphi(\tau)]^{2}-[\bar{H}_{z}]^{2}\right)\end{aligned} \tag{55}$$

4. Numerical results and discussion

In order to study the effect of Rosenthal heat source and eddy current loss under the influence of magnetic field in a conducting rod, we considered copper material for numerical computations and values of material constant are taken as [5]

$$\mu=4\pi\times10^{-7}\left[\frac{H}{m}\right], \sigma=3.42\times10^{7}\left[\frac{S}{m}\right], \kappa=92.6\times10^{-6}\left[\frac{m^{2}}{sec}\right],$$
$$\nu=0.33, E=70[GPa], K=386Wm^{-1}k^{-1}, \alpha_{t}=1.78\times10^{-5}[1/K],$$
$$\chi_{1}=3.98\times10^{-3}$$

The arbitrary excitation function $\varphi(\tau)$ and the function $\bar{f}_{1}(\tau)$ are chosen in the following form.

$$\bar{f}_{1}(\tau)=e^{\tau}, \quad \phi(\tau)=1-e^{\tau} \tag{56}$$

To illustrate the magneto-thermoelastic problem with time fractional temperature distribution we keep the speed of moving velocity at $\vartheta=2.4$. We consider numerical computation which gives the variation in magnetic field, eddy current loss, temperature change and stress field for the various values of fractional parameter (α). Figure 1 represents variation of magnetic field with time for various values of α = 0.25,0.75,1.25,1.5. It is found that the magnetic field increases and become steady as time passes. In Figure 2, the distribution of eddy current loss with time for the different values of α are plotted. Eddy current shows peak near about $\tau=0.18$ and it decays slowly with time. Figure 3 and Figure 4 represent temperature distribution with time for the various values of α and ϑ. The effect of eddy current loss and Rosenthal heat source on temperature gives peak for different values of α and velocity. Temperature shows peak at time $\tau=0.44$.

Figure 5 shows temperature distribution with space variable x for the value of $\vartheta=2.4$ and $t=0.1$ such that the peak arrive at $x=0.24$ and it is also observed that the temperature gradually decreases as x increases. The stress is a sum of two components; one is thermal stress and magnetic stress. Figure 6 and Figure 7 represents thermal stress distribution with time and shows significant changes with the passage of time. While the Figure 8, Figure 9 and Figure 10 represents magnetic stresses with respect to time. It shows the significant difference for fractional parameter.

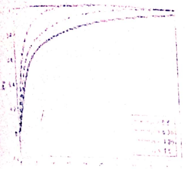

Fig. 1. Distribution of magnetic field (H_z) verses time (τ) for various values of α.

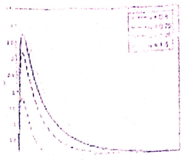

Fig2. Distribution of eddy current loss (ω) verses time (τ) for various values of α

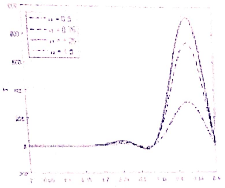

Fig. 3. Distribution of temperature (T) verses time (τ) for various values of α

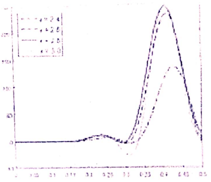

Fig. 4. Distribution of temperature (T) verses time (τ) for various values of [illegible]

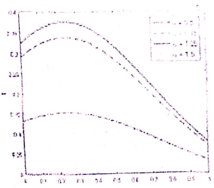

Fig. 5. Distribution of temperature (T) verses time (x) for various values of α

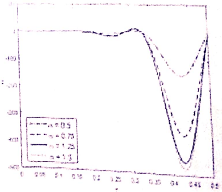

Fig. 6. Distribution of thermal stress (σ_{yy}^T) verses time (τ) for various values of α

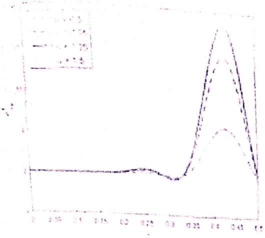

Fig. 7. Distribution of thermal stress (σ_{xx}^T) verses time (τ) for various values of α

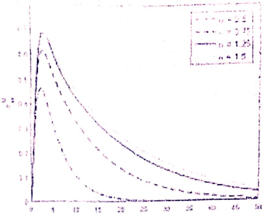

Fig. 8. Distribution of magnetic stress (σ_{xx}^M) verses time (τ) for various values of α

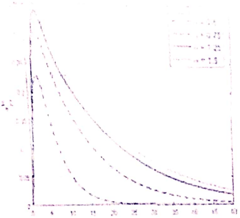

Fig. 9. Distribution of magnetic stress (σ_{xx}^M) verses time (τ) for various values of α

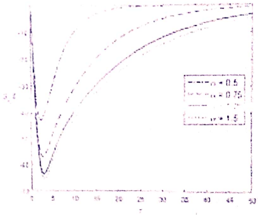

Fig. 10. Distribution of magnetic stress (σ_{xx}^M) verses time (τ) for various values of α

5 Conclusion

In this paper, a conducting rod under the effect of magnetic field due to eddy current loss with the time fractional thermoelasticity theory is presented. The heat conduction equation is subjected to Rosenthal heat source. The problem is solved by using Laplace and Fourier transform technique. The effect of different values of fractional parameter α on temperature and stresses are observed. We conclude the following facts.

1. It is clear that in a conducting rod, in all figures, the distributions considered have nonzero value which means that the region does not felt a thermal disturbance.
2. The increase in time fractional order parameter, the temperature and the eddy current loss increases while the effect of magnetic field and the stress components $\overline{\sigma_{yy}}^T(\bar{x},\tau)$, $\overline{\sigma_{yy}}^M(\bar{x},\tau)$ reduces. On the other hand, the remaining components of stresses are also reduces with the decrease in fractional order.
3. The temperature of the conducting rod increases as the speed of the moving heat source increases.

The result of this analysis is expected to help in understanding the phenomena of magneto-thermoelasticity development in conducting rod of fractional order parameters.

Acknowledgements

The first author wishes to thank, Dr Ganesh Kedar of Rashtrasant Tukdoji Maharaj Nagpur Uni- versity, Nagpur for their assistance that greatly improved the manuscript. Also wishes to thank the refrees for their valuable comments and suggestions.

References

[1] V. S. Kulkarni and K. C. Deshmukh, "A Brief Note on Quasistatic Thermal Stresses in a Rectangular Plate" Far East Journal of Applied Mathematics, 26(1):349–360, October 2007.

[2] K. C. Deshmukh and M. V. Khandait., "Thermal Stresses in a Simply Supported Plate with Thermal Bending Moments with Heat Source" Material Physics and Mechanics, 21(2):135–146, August 2014. ISSN 0022-1481. doi: 10.1115/1.4035442.

[3] R. L. Stoll. "The Analysis of Eddy Currents", Clarendon Press Oxford, 26(4):124–126, Decem- ber 1974. URL http://adsabs.harvard.edu/abs/2018arXiv180100775H.

[4] F. C. Moon and S. Chattopadhyay. "Magnetically Induced Stress Waves in a Conducting Solid", Theory and Experiment. Journal of Applied Mechanics, 41(3):641–647, September 1974. doi: 10.1115/1.3423363.

[5] M. Higuchi, R. Kawamura, Y. Tanigawa, and H. Fujieda "Magnetothermoelastic Stresses Induced by a Transient Magnetic Field in an Infinite Conducting Plate". Journal of Mechanics Of Material And Structures, 2(1):113–130, March 2007. doi: 10.2140/jomms.2007.2.113.

[6] Daniel Rosenthal. "The theory of moving sources of heat and its application of metal treatments". Transactions of ASME, 68 849–866, 1946.

[7] R Weichert and K Scho"nert., "Temperature distribution produced by a moving heat source". The Quarterly Journal of Mechanics and Applied Mathematics, 31(3):363–379, 1978.

[8] Xuan-Tan Pham., "Two-dimensional rosenthal moving heat source analysis using the meshless element free galerkin method". Numerical Heat Transfer, Part A: Applications, 63(11):807–823, 2013.

[9] Kenneth S Miller and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations". Wiley, 1993.

[10] Tianhu He and Ying Guo., "A one-dimensional thermoelastic problem due to a moving heat source under fractional order theory of thermoelasticity". Advances in Materials Science and Engineering, 2014.

[11] Rudolf Gorenflo, Asaf Iskenderov, and Yuri Luchko. "Mapping between solutions of fractional diffusion-wave equations". Fractional Calculus and Applied Analysis, 3(1):75–86, 2000.

[12] Y.Z. Povstenko. "Theory of thermoelasticity based on the space-time-fractional heat conduction equation". Physica Scripta, 2009(T136):014017, 2009.

[13] A.A. Kilbas and Srivastava HM& Trujillo JJ. 2006 "theory and applications of fractional differ ential equations.

[14] Igor Podlubny. "Fractional differential equations", vol. 198 of mathematics in science and en- gineering, 1999.

[15] Alexander I. Saichev and George M Zaslavsky. Fractional kinetic equations: solutions and applications. Chaos: An Interdisciplinary Journal of Nonlinear Science, 7(4):753–764, 1997.

[16] N. Noda, RB Hetnarski, and Y Tanigawa. Thermal stresses, taylor & francis. New York, 260, 2003.

[17] J.J.Bikram, G.D. Kedar, V. Raut,"Study of Time fractional heat conduction equation in rectangular plate. Novyi MIR Research journal, 5(10) 93-104,2020.

[18] J.J.Bikram, G.D. Kedar "Study of Thermoelasticity Problem based on Space Fractional Heat Conduction Equation by HPM", Science and Engineering Journal, 24(10): 387 - 394,2020.

[19] S. P.Pawar, J.J.Bikram, G.D. Kedar ," Thermoelastic Behaviour in a Multilayer Composite Hollow Sphere with Heat Source" ornal of Solid mechanics,12(4) 883-901,2020.